$$C^t\begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ 0 & 0 \ldots\ldots 0 \end{Bmatrix} = \sum_{0}^{j_h} {}_{\alpha\ \beta \ldots\ldots \kappa}\, C^{t-j_h}\begin{Bmatrix} s_1 & s_2 \ldots\ldots s_k \\ \alpha & \beta \ldots\ldots \kappa \end{Bmatrix}.$$

This is the relation between coefficients in the process of multiplication.

3. If we have a table giving the product of $|\,0\,1\,2\,3\ldots\ldots(n-1)\,|$ by all the symmetric functions of weight w, of the form (2), we can by means of the relation between coefficients found in article 2 construct a table giving the product of $|\,0\,1\,2\,3\ldots\ldots(n-1)\,|$ by all symmetric functions of the same type of weight $w+j_h$.

The following tables of this kind from weight one to weight seven have been constructed by means of the relation of article 2.

In this way we can construct the table of order t having those of order less than t.

	$\|0123\cdots(n-2)n\|$	$\|0123\cdots(n-3)(n-1)n\|$	$\|0123\cdots(n-2)(n-1)n\|$	$\|0123\cdots(n-4)(n-2)(n-1)n\|$	$\|0123\cdots(n-3)(n-2)(n-1)n\|$	$\|0123\cdots(n-2)^2(n-1)n\|$	$\|0123\cdots(n-3)^2(n-1)n\|$	$\|0123\cdots(n-2)^2(n-1)^2n\|$	$\|0123\cdots(n-5)(n-3)(n-2)(n-1)n\|$	$\|0123\cdots(n-4)(n-2)^2(n-1)n\|$	$\|0123\cdots(n-3)^2(n-2)(n-1)n\|$	$\|0123\cdots(n-4)(n-3)(n-1)^2n\|$	$\|0123\cdots(n-3)^2(n-2)^2n\|$	$\|0123\cdots(n-4)(n-3)^2(n-2)n\|$	$\|0123\cdots(n-5)(n-3)(n-2)^2(n-1)n\|$	$\|0123\cdots(n-4)(n-3)(n-2)(n-1)^2n\|$	$\|0123\cdots(n-3)^2(n-2)(n-1)^2n\|$	$\|0123\cdots(n-4)(n-3)^2(n-2)^2n\|$	$\|0123\cdots(n-3)^2(n-2)^2(n-1)^2n\|$
$\sum a_1$	1																		
$(\sum a_1)^2$		1	1																
$\sum a_1 a_2$		1																	
$(\sum a_1)^3$				1	2	1													
$\sum a_1 a_2 \sum a_1$				1	1														
$\sum a_1 a_2 a_3$				1															
$(\sum a_1)^4$							2	1	3	3	1								
$\sum a_1 a_2 (\sum a_1)^2$							1	1	2	1									
$(\sum a_1 a_2)^2$							1	1	1										
$\sum a_1 a_2 a_3 \sum a_1$								1	1										
$\sum a_1 a_2 a_3 a_4$								1											
$(\sum a_1)^5$												5	5	1	4	6	4	1	
$\sum a_1 a_2 (\sum a_1)^3$												2	3	1	3	3	1		
$(\sum a_1 a_2)^2 \sum a_1$												1	2	1	2	1			
$\sum a_1 a_2 a_3 (\sum a_1)^2$													1	1	2	1			
$\sum a_1 a_2 a_3 \sum a_1 a_2$													1	1	1				
$\sum a_1 a_2 a_3 a_4 \sum a_1$														1	1				
$\sum a_1 a_2 a_3 a_4 a_5$														1					

	$\frac{1}{0!1!2!3}(n-4)(n-1)n(n+1)$	$\frac{1}{0!1!2!3}(n-5)(n-3)(n-2)n(n+1)$	$\frac{1}{0!1!2!3}(n-4)(n-2)n(n+2)$	$\frac{1}{0!1!2!3}(n-3)n(n+3)$	$\frac{1}{0!1!2!3}(n-5)(n+1)(n+2)$	$\frac{1}{0!1!2!3}(n-7)(n-4)(n-1)(n+2)$	$\frac{1}{0!1!2!3}(n-6)(n-4)(n-1)(n+3)$	$\frac{1}{0!1!2!3}(n-5)(n-3)(n-2)(n+3)$	$\frac{1}{0!1!2!3}(n-4)(n-2)(n-1)(n+4)$	$\frac{1}{0!1!2!3}(n-3)(n-1)(n+5)$
$(\Sigma a_1)^6 a$	5	9	16	9	5	1	5	10	10	5	1
$\Sigma a_1 a_2 (\Sigma a)^4$	3	6	8	3	2	1	4	6	4	1	
$(\Sigma a_1 a_2)^2 (\Sigma a_1)^2$	2	4	4	1	1	1	3	3	1		
$(\Sigma a_1 a_2)^3$	1	3	2		1	1	2	1			
$\Sigma a_1 a_2 a_3 (\Sigma a_1)^3$	1	3	2			1	3	3	1		
$\Sigma a_1 a_2 \Sigma a_1 a_3 \Sigma a$	1	2	1			1	2	1			
$(\Sigma a_1 a_2 a_3)^2$	1	1				1	1				
$\Sigma a_1 a_2 a_3 a_4 (\Sigma a)^2$		1				1	2	1			
$\Sigma a_1 a_2 a_3 a_4 \Sigma a_1 a_2$		1				1	1				
$\Sigma a_1 a_2 a_3 a_4 a_5 \Sigma a_1$						1	1				
$\Sigma a_1 a_2 a_3 a_4 a_5 a_6$						1					

	$\frac{1}{0!1!2!3}(n-5)(n-3)(n-1)n(n+1)$	$\frac{1}{0!1!2!3}(n-6)(n-4)(n-3)(n-2)n(n+1)$	$\frac{1}{0!1!2!3}(n-4)(n-3)(n-1)n(n+2)$	$\frac{1}{0!1!2!3}(n-5)(n-3)(n-2)n(n+2)$	$\frac{1}{0!1!2!3}(n-4)(n-2)n(n+3)$	$\frac{1}{0!1!2!3}(n-4)(n-2)(n+1)(n+2)$	$\frac{1}{0!1!2!3}(n-3)n(n+4)$	$\frac{1}{0!1!2!3}(n-3)(n+1)(n+3)$	$\frac{1}{0!1!2!3}(n-1)n$	$\frac{1}{0!1!2!3}(n-8)(n-6)-(n-1)$	$\frac{1}{0!1!2!3}(n-7)(n-5)-(n-1)(n+1)$	$\frac{1}{0!1!2!3}(n-6)(n-4)-(n-1)(n+2)$	$\frac{1}{0!1!2!3}(n-5)(n-3)-(n-1)(n+3)$	$\frac{1}{0!1!2!3}(n-4)(n-2)-(n-1)(n+4)$	$\frac{1}{0!1!2!3}(n-3)(n-1)(n+5)$	$\frac{1}{0!1!2!3}(n-2)(n+6)$
$(\Sigma a_1)^7$	14	14	21	35	35	21	14	14	1	6	15	20	15	6	1	
$\Sigma a_1 a_2 (\Sigma a_1)^5$	9	10	11	20	15	10	4	5	1	5	10	10	5	1		
$(\Sigma a_1 a_2)^2 (\Sigma a_1)^3$	6	7	6	11	6	5	1	2	1	4	6	4	1			
$(\Sigma a_1 a_2)^3 \Sigma a_1$	4	5	3	6	2	3		1	1	3	3	1				
$\Sigma a_1 a_2 a_3 (\Sigma a_1)^4$	4	6	3	8	3	2			1	4	6	4	1			
$\Sigma a_1 a_2 a_3 \Sigma a_1 a_2 (\Sigma a_1)^3$	3	4	2	4	1	1			1	3	3	1				
$\Sigma a_1 a_2 a_3 (\Sigma a_1 a_2)^2$	2	3	1	2		1			1	2	1					
$(\Sigma a_1 a_2 a_3)^2 \Sigma a_1$	2	2	1	1					1	2	1					
$\Sigma a_1 a_2 a_3 a_4 (\Sigma a_1)^3$	1	3		2					1	3	3	1				
$\Sigma a_1 a_2 a_3 a_4 \Sigma a_1 a_2 \Sigma a_1$	1	2		1					1	2	1					
$\Sigma a_1 a_2 a_3 a_4 \Sigma a_1 a_2 a_3$	1	1							1	1						
$\Sigma a_1 a_2 a_3 a_4 a_5 (\Sigma a_1)^2$		1							1	2	1					
$\Sigma a_1 a_2 a_3 a_4 a_5 \Sigma a_1 a_2$		1							1	1						
$\Sigma a_1 a_2 a_3 a_4 a_5 a_6 \Sigma a_1$									1	1						
$\Sigma a_1 a_2 a_3 a_4 a_5 a_6 a_7$									1							

4. If we denote the product

$$|\,0123\ldots(n-1)\,|\,\Big(\sum a_1a_2a_3\ldots a_{j_2}\Big)^{i_2}\Big(\sum a_1a_2a_3\ldots a_{j_3}\Big)^{i_3}\ldots\Big(\sum a_1a_2a_3\ldots a_{j_k}\Big)^{i_k-1}$$

by P, then the alternant

$|\,0123\ldots r_1s_1\ldots r_2s_2\ldots r_gs_g\ldots r_ks_k\,|$ —which we shall represent temporarily by A_1— in the product $P\Sigma a_1a_2a_3\ldots a_{j_1}$, (or P_1), can arise from the following alternants of P:

$$|\,0\,1\,2\,3\ldots r_1(s_1-1)s_1(s_1+1)\ldots(s_1+\alpha_1-2)(s_1+\alpha_1)\ldots r_g(s_g-1)(s_g)\ldots$$

$$\ldots(s_g+\gamma_1-2)(s_g+\gamma_1)\ldots r_k(s_k-1)\,|\,,$$

where $\alpha_1, \beta_1, \ldots\gamma_1\ldots\kappa_1=0, 1, 2, 3\ldots j_1$,

and $\alpha_1+\beta_1+\gamma_1\ldots\kappa_1=j_1$.

The alternant

$$|\,0\,1\,2\,3\ldots(r_1-j_h+1)\ldots r_1(r_1+1)s_1\ldots r_2s_2\ldots r_g\,s_g\,\ldots r_ks_k\,|\,,$$

which we shall also denote temporarily by A_2 in the product $P.\Sigma a_1a_2a_3\ldots a_{j_1+j_h}$, (or P_2), will evidently arise from the same terms of P, but it will arise also from the following terms:

$$|\,0\,1\,2\,3\ldots(r_1-j_h+1)\ldots r_1(r_1+1)s_1(s_1+1)\ldots(s_1+\alpha+\alpha_1-2)(s_1+\alpha+\alpha_1)\ldots$$

$$\ldots r_g\,s_g\,(s_g+1)\ldots(s_g+\alpha+\alpha_1-2)(s_g+\alpha+\alpha_1)\ldots r_k(s_k-1)\,|\,.$$

These terms could not exist, however, if $j_1\overline{\geq}\frac{1}{2}n$.

5. If we denote the coefficient of A_2 by

$$C^t{}_{r_1}^{j_h}\left\{\begin{matrix}s_1 & s_2\ldots s_k\\ 0 & 0\ldots 0\end{matrix}\right\}$$

where the j_h above the r_1 denotes that r_1 and the preceding (j_h-1) numbers are increased by unity.

Then, if $j_1\overline{\geq}\frac{1}{2}n$, we will have

$$C^t{}_{r_1}^{j_h}\left\{\begin{matrix}s_1 & s_2\ldots s_k\\ 0 & 0\ldots 0\end{matrix}\right\}=\sum_0^{j_h}{}_{\alpha\,\beta\ldots\kappa}\,C^{t-j_h}{}_{r_1}^{j_h}\left\{\begin{matrix}s_1 & s_2\ldots s_k\\ \alpha & \beta\ldots\kappa\end{matrix}\right\};$$

for, under these conditions the coefficient of A_1 in the product P_1 is the same as the coefficient of A_2 in P_2.

This gives us a means of obtaining the coefficient of any alternant in the product $P_1 . \Sigma a_1 a_2 a_3 \ldots a_{j_k}$, provided we have all those in the product P_2; that is, we obtain coefficients in this table from others in the same table.

6. Having an incomplete table as is considered in the preceding article we could readily convert it into a complete table of order $(t-j_1)$ by removing in all possible ways j_1 numbers from the alternants in the product.

7. Let us now consider, in the product P_1:

$$| 0\ 1\ 2\ 3 \ldots (n-1) | \sum a_1 a_2 a_3 \ldots a_{n-i} \left(\sum a_1\right)^i,$$

those alternants of the type

$$| 0\ 1\ 2\ 3 \ldots r_1 s_1 \ldots r_2 s_2 \ldots r_3 s_3 | ,$$

where $s_1 - r_1 = 2$, $s_2 - r_2 = 2$, and $s_3 - r_3 = q$.

This alternant in P_1 can evidently arise on multiplying

$$| 0\ 1\ 2\ 3 \ldots r_1 s_1 \ldots\ldots r_2 s_2 \ldots\ldots r_3 (s_3 - 1) |$$
$$| 0\ 1\ 2\ 3 \ldots r_1 s_1 \ldots\ldots r_2 (s_2 - 1)(s_2 + 1) \ldots r_3 s_3 |$$
$$| 0\ 1\ 2\ 3 \ldots r_1 (s_1 - 1)(s_1 + 1) \ldots r_2 s_2 \ldots\ldots r_3 s_3 |$$

by Σa_1, in the product

$$| 0\ 1\ 2\ 3 \ldots (n-1) | \sum a_1 a_2 a_3 \ldots a_{n-i} \left(\sum a_1\right)^{i-1},$$

so that we have

$$C^i \begin{Bmatrix} s_1 & s_2 & s_3 \\ 0 & 0 & 0 \end{Bmatrix} = \sum_0^1 \alpha\beta\gamma\, C^{i-1} \begin{Bmatrix} s_1 & s_2 & s_3 \\ \alpha & \beta & \gamma \end{Bmatrix} \quad (\alpha+\beta+\gamma=1).$$

8. Since $n-1 \geqslant \frac{1}{2}n$, the conditions of articles 4 and 5 are satisfied and we have the following reduction formula for the construction of tables:

$$C^i {}^{0}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix} = C^{i-1} {}^{1}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 1 \end{Bmatrix} + C^{i-1} {}^{1}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 1 & 0 \end{Bmatrix} + C^{i-1} {}^{0}_{s_1} \begin{Bmatrix} s_2 & s_3 \\ 0 & 0 \end{Bmatrix}.$$

In the following table

$$\overline{\sum} a_1 = | 0\ 1\ 2\ 3 \ldots (n-1) | \sum a_1 a_2 a_3 \ldots a_{n-1} \left(\sum a_1\right)',$$
$$\left(\overline{\sum} a_1\right)^2 = | 0\ 1\ 2\ 3 \ldots (n-1) | \sum a_1 a_2 a_3 \ldots a_{n-2} \left(\sum a_1\right)^2,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\left(\overline{\sum} a_1\right)'' = | 0\ 1\ 2\ 3 \ldots (n-1) | \sum a_1 a_2 a_3 \ldots a_{n-11} \left(\sum a_1\right)''.$$

	1123·· n!	1023· (n-1)n(n+1)!	1013· (n-2)n(n+1)!	1013· (n-1)n(n+2)!	10124 (n-3)(n-1) n(n+1)!	10124 (n-2) n(n+2)!	10124 (n-1)(n+3)!	101235 (n-4)(n-2)(n-1) n(n+2)!	101235 (n-3)(n-1) n(n+2)!	101235 (n-2) n(n+3)!	101235 (n-1)(n+4)!	101··46 (n-5)n(n-3)·· n(n+1)!	101··46··(n-4)(n-2)(n-1) n(n+2)!	101··46··(n-3)(n-1)n(n-3)!	101··46·(n-2)n(n+4)!	101··46··(n-1)(n+5)!	101··57··(n-6)(n-4)·· n(n+1)!	101··57··(n-5)(n-3)·· n(n+2)!	101··57··(n-4)(n-2)(n-1) n(n+3)!	101··57··(n-3)(n-1) n(n+4)!	101··57·· (n-2)n(n+5)!	101··57·· (n-1)(n+6)!	101···68··(n-7)(n-5)·· n(n+1)!	101···68··(n-6)(n-4)·· n(n+2)!	101···68··(n-5)(n-3)·· n(n+3)!	101···68··(n-4)(n-2)(n-1) n(n+4)!	101···68··(n-3)(n-1) n(n+5)!	101···68··(n-2) n(n+6)!	101···68·(n-1)(n+7)!
$(\Sigma a_1)^1$	1	1																											
$(\Sigma a_1)^2$	1	2	1	1																									
$(\Sigma a_1)^3$	1	3	3	3	1	2	1																						
$(\Sigma a_1)^4$	1·	4	6	6	4	8	4	1	3	3	1																		
$(\Sigma a_1)^5$	1	5	10	10	10	20	10	5	15	15	5	1	4	6	4	1													
$(\Sigma a_1)^6$	1	6	15	15	20	40	20	15	45	45	15	6	24	36	24	6	1	5	10	10	5	1							
$(\Sigma a_1)^7$	1	7	21	21	35	70	35	35	105	105	35	21	84	126	84	21	7	35	70	70	35	7	1						
$(\Sigma a_1)^8$	1	8	28	28	56	112	56	70	210	210	70	56	224	336	224	56	28	140	280	280	140	28	8	48	120	160	120	48	8
$(\Sigma a_1)^9$	1	9	36	36	84	168	84	126	378	378	126	126	504	756	504	126	84	420	840	840	420	84	36	216	540	720	540	216	36
$(\Sigma a_1)^{10}$	1	10	45	45	120	240	120	210	630	630	210	252	1008	1512	1008	252	210	1050	2100	2100	1050	210	120	720	1800	2400	1800	720	120
$(\Sigma a_1)^{11}$	1	11	55	55	165	330	165	330	990	990	330	462	1848	2772	1848	462	462	2310	4620	4620	2310	462	330	1980	4950	6600	4950	1980	330

9. We wish to prove the formula

$$C^{i\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=\frac{i(i-1)\ (i-2)\ldots\ldots(i-s_1+2)}{\underline{|s_1-1}}\cdot\frac{(s_1-2)(s_2-3)\ldots\ldots(s_1-s_3+r_3-2)}{\underline{|s_3-r_3-1}}.$$

In the reduction formula of article 8, let i receive the successive values 2, 3, 4, 5.....i, and we have the following result:

$$C^{2\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{1\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{1\,1}_{s_1}\left\{\begin{matrix}s_1 & s_2\\ 1 & 0\end{matrix}\right\}+C^{1\,0}_{s_1}\left\{\begin{matrix}s_2 & s_2\\ 0 & 0\end{matrix}\right\}$$

$$C^{3\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}+C^{2\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}$$

$$C^{4\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{3\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{3\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}+C^{3\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}$$

$$C^{5\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{4\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{4\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}+C^{4\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}$$

....

$$C^{i-1\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{i-2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{i-2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_2\\ 1 & 0\end{matrix}\right\}+C^{i-2\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}$$

$$C^{i\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{i-1\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{i-1\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}+C^{i-1\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}$$

$$C^{i\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=\sum_1^{i-1}{}^{j}\ C^{j\,1}_{s_1}\left\{\begin{matrix}s_2 & s_2\\ 0 & 1\end{matrix}\right\}+\sum_1^{i-1}{}^{j}\ C^{j\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}$$

10. Let the reduction formula, designated R,

$$C^{1\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=\sum_1^{i-1}{}^{j}\ C^{j\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+\sum_1^{i-1}{}^{j}\ C^{j\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}$$

operate upon itself and the successive results:

$$C^{i\,0}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 0\end{matrix}\right\}=C^{i-1\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{i-1\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}$$

$$C^{i-2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{i-2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}$$

$$C^{i-3\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{i-3\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}$$

....

$$C^{s_1-2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 1\end{matrix}\right\}+C^{s_1-2\,1}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 0\end{matrix}\right\}$$

Observe that the terms of R beyond the limit s_1-2 are zero.

$$=1\Big(C^{i-2\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 0\end{matrix}\right\}+2\,C^{i-2\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 1\end{matrix}\right\}+C^{i-2\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 2\end{matrix}\right\}\Big)$$

$$2\Big(C^{i-3\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 0\end{matrix}\right\}+2\,C^{i-3\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 1\end{matrix}\right\}+C^{i-3\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 2\end{matrix}\right\}\Big)$$

(*a*) $$3\Big(C^{i-4\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 0\end{matrix}\right\}+2\,C^{i-4\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 1\end{matrix}\right\}+C^{i-4\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 2\end{matrix}\right\}\Big)$$

....

($i-2$) $$C^{s_1-3\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 0\end{matrix}\right\}+2\,C^{s_1-3\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 1\end{matrix}\right\}+C^{s_1-3\,2}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 2\end{matrix}\right\}$$

$$=1\Big(C^{i-3\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 3 & 0\end{matrix}\right\}+3C^{i-3\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 1\end{matrix}\right\}+3C^{i-3\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 2\end{matrix}\right\}+C^{i-3\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 3\end{matrix}\right\}\Big)$$

$$3\Big(C^{i-4\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 3 & 0\end{matrix}\right\}+3C^{i-4\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 1\end{matrix}\right\}+3C^{i-4\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 2\end{matrix}\right\}+C^{i-4\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 3\end{matrix}\right\}\Big)$$

(*b*) $$6\Big(C^{i-5\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 3 & 0\end{matrix}\right\}+3C^{i-5\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 1\end{matrix}\right\}+3C^{i-5\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 2\end{matrix}\right\}+C^{i-5\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 3\end{matrix}\right\}\Big)$$

$$10\Big(C^{i-6\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 3 & 0\end{matrix}\right\}+3C^{i-6\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 2 & 1\end{matrix}\right\}+3C^{i-6\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 1 & 2\end{matrix}\right\}+C^{i-6\,3}_{s_1}\left\{\begin{matrix}s_2 & s_3\\ 0 & 3\end{matrix}\right\}\Big)$$

....

$$\frac{(i-2)(i-3)}{\lfloor 2}\Big(C^{s_1-4\,3}_{s_1}\begin{Bmatrix}s_2\,s_3\\3\,0\end{Bmatrix}+3C^{s_1-4\,3}_{s_1}\begin{Bmatrix}s_2\,s_3\\2\,1\end{Bmatrix}$$

$$+3C^{s_1-4\,3}_{s_1}\begin{Bmatrix}s_2\,s_3\\1\,2\end{Bmatrix}+C^{s_1-4\,3}_{s_1}\begin{Bmatrix}s_2\,c_3\\0\,3\end{Bmatrix}\Big)$$

....

....

$$=1\Big(C^{i-s_1+2\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-2\,0\end{Bmatrix}+(s_1-2)\,C^{i-s_1+2\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-3\,1\end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\lfloor 2}C^{i-s_1+2\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-4\,2\end{Bmatrix}+\ldots\Big)$$

$$(s_1-2)\Big(C^{i-s_1+1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-2\,0\end{Bmatrix}+(s_1-2)\,C^{i-s_1+1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-3\,1\end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\lfloor 2}C^{i-s_1+1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-4\,2\end{Bmatrix}+\ldots\Big)$$

$$(c)\quad\frac{(s_1-2)(s_1-3)}{\lfloor 2}\Big(C^{i-s_1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-2\,0\end{Bmatrix}+(s_1-2)\,C^{i-s_1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-3\,1\end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\lfloor 2}C^{i-s_1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-4\,2\end{Bmatrix}+\ldots\Big)$$

$$\frac{(s_1-2)\,(s_1-3)\,(s_1-4)}{\lfloor 3}\Big(C^{i-s_1-1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-2\,0\end{Bmatrix}$$

$$+(s_1-2)\,C^{i-s_1-1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-3\,1\end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\lfloor 2}C^{i-s_1-1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-4\,2\end{Bmatrix}+\ldots\Big)$$

....

$$\frac{(i-2)\,(i-3)\ldots(i-s_1+2)}{\lfloor s_1-3}\Big(C^{1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-2\,0\end{Bmatrix}+(s_1-2)\,C^{1\,s_1-1}_{s_1}\begin{Bmatrix}s_2\,s_3\\s_1-3\,1\end{Bmatrix}$$

$$+\frac{(s_1-2)(s_1-3)}{\lfloor 2}\,C^{1s_1-1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-4 & 2\end{Bmatrix}+\ldots\ldots+\ldots\ldots$$

$$\ldots\ldots+\frac{(s_1-2)(s_1-3)\ldots\ldots(s_1-k-1)}{\lfloor k}\,C^{1s_1-1}_{s_1}\begin{Bmatrix} s_2 & s_3 \\ s_1-k-2 & k\end{Bmatrix}+\ldots\ldots\Big).$$

In (c) let $i=i$, $s_1=s_1$, and $s_3=r_3+1$. Then

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ 0 & 0\end{Bmatrix}=1\,C^{i-s_1+2\,s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_1-2 & 0\end{Bmatrix}+$$

$$(s_1-2)\,C^{i-s_1+1\,s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_1-2 & 0\end{Bmatrix}+\frac{(s_1-1)(s_1-2)}{\lfloor 2}\,C^{i-s_1\,s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_2-1 & 0\end{Bmatrix}$$

$$+\ldots\ldots+\frac{(i-2)\,(i-3)(i-4)\ldots\ldots(i-s_1+2)}{\lfloor s_1-3}\,C^{1s_1-1}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ s_1-1 & 0\end{Bmatrix}.$$

It is evident from the table that the C's have values, respectively, from $i-s_1+2$ down to 1. Wherefore

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ 0 & 0\end{Bmatrix}=1.(i-s_1+2)+(s_1-2)(i-s_1+1)+\frac{(s_1-1)(s_1-2)}{\lfloor 2}(i-s_1)$$

$$+\frac{s_1(s_1-1)(s_1-2)}{\lfloor 3}(i-s_1-1)+\ldots\ldots+\frac{(i-2)(i-3)(i-4)\ldots\ldots(i-s_1+2)}{\lfloor s_1-3}(i-(i-1))$$

$$=i\Big[1+(s_1-2)+\frac{(s_1-1)(s_1-2)}{\lfloor 2}+\frac{s_1(s_1-1)(s_1-2)}{\lfloor 3}+$$

$$\ldots\ldots+\frac{(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-3}\Big]-(s_1-2)\Big[1+(s_1-1)$$

$$+\frac{(s_1-1)s_1}{\lfloor 2}+\frac{(s_1-1)s_1(s_1+1)}{\lfloor 3}+\ldots\ldots+\frac{(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1+2}\Big].$$

Summing the two series in the brackets, we have

$$C^{i\,0}_{s_1}\begin{Bmatrix} s_2 & (r_3+1) \\ 0 & 0\end{Bmatrix}=\frac{i(i-1)\,(i-2)\ldots\ldots\,(i-s_1+2)}{\lfloor s_1-2}-(s_1-2)\frac{i(i-1)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}$$

$$= \frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1} (s_1-1-(s_1-2))$$

$$= \frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1} .1.$$

In a similar manner, it can be proven that

$$C^{i}_{s_1} \begin{Bmatrix} 0 & s_2 & r_3+2 \\ 0 & & 0 \end{Bmatrix} = \frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1} (s_1-2)$$

$$C^{i}_{s_1} \begin{Bmatrix} 0 & s_2 & r_3+3 \\ 0 & & 0 \end{Bmatrix} = \frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1} \frac{(s_1-2)\ (s_1-3)}{\lfloor 2},$$

or, in general,

$$C^{i}_{s_1} \begin{Bmatrix} 0 & s_2 & s_3 \\ 0 & & 0 \end{Bmatrix} = \frac{i(i-1)(i-2)(i-3)\ldots\ldots(i-s_1+2)}{\lfloor s_1-1}$$

$$\times \frac{(s_1-2)\ (s_1-3)\ldots\ldots(s_1-s_3+r_3-2)}{\lfloor s_3-r_3-1}.$$

SYRACUSE UNIVERSITY, *December 18, 1902.*

AN ACCOUNT OF PROFESSOR RUNKLE'S MATHEMATICAL MONTHLY.

By PROFESSOR SIMON NEWCOMB.

I first made Mr. Runkle's acquaintance in the winter of 1857, when he was senior assistant in the Nautical Almanac Office, then at Cambridge, Mass. His intelligence, intellectual activity, and lively interest in matters and things generally, not excluding things political, made him a very interesting character.

It was early in 1858 that he announced to me and some others in the office his intention of starting a mathematical journal. His first step was to secure the necessary support. It may be feared that few in our day have an adequate conception of the backward condition of mathematical study in our country at that time. A curious illustration is offered by Davies' well-known dictionary of mathematics, in the preface of which it was announced that it contained defin-

itions of all the terms used in mathematics. The word "determinant" does not, however, occur in the dictionary, nor any terms used in the branches of algebra with which determinants would be associated. So far as I know, the first person in our country to know that a determinant existed, or to become acquainted with a paper in any European mathematical journal, was Prof. Benj. Peirce. Cayley and Sylvester were then at the height of their activity in developing the newer theories, but Peirce was the only one among us who had the slightest idea what they were working at.

Such was the field in which Runkle proposed to sow the seeds of what might be considered a new branch of learning. His circulars were sent to all the leading mathematical professors in the country and others interested in the subject. The responses were such as to justify his going on with the undertaking, although the problem of making a respectable standard compatible with the necessary pecuniary support was a very doubtful one.

The first number was issued in January, 1859, and showed an attempt to compromise between the two difficulties. Its most marked feature, and one which seemed admirably adapted to make it conducive to the advancement of mathematical science, was the prize problem for students. These were of various degrees of simplicity, though all were above the school-boy standard. A first prize was offered for the best solution, and a second prize for the next best. Prize essays were also invited. One of the problems of the first number, which related to a theorem regarding the intersection of circles, brought out a short paper from Cayley, in which he extended the theorem to conics.

The Memoir which won the first prize was by G. W. Hill, on a subject relating to the theory of the figure of the earth. It was almost his first paper published, but showed unmistakably the hand of the master.

The prizes were awarded on the recommendation of a committee to whom the solutions were submitted. Professor Winlock was the first chairman of the committee, but soon retired from it, and the solutions were afterwards passed upon by W. P. G. Bartlett and myself.

Bartlett was a man whose untimely death was a great loss for mathematical science in our country. He was not only of the first order of ability, but an enthusiastic devotee of mathematical study, whose character and standard were in every respect the highest. He was a nephew of the eminent lawyer, Mr. Sidney Bartlett, long a leader of the Boston bar. One day when his uncle urged him to undertake the profession of law in order to secure wealth and position in society, he replied: "If I can secure a high position in the regard of Pierce, Cayley, Sylvester, and fifteen or twenty other men like them, I shall value it more than any opinion that society at large can form of me."

Although not formally connected with the responsible editorship, Bartlett and myself naturally acted as general advisers and referees on matters connected with the Journal.

Among our amusing experiences was one with a Pennsylvania politician of prominence in the State, who had written and read before a Teachers' Associa-

tion a long and rambling essay on the then celebrated problem of the oxen and the field of grass. Not knowing what to do with the paper, the Association referred it to the editor of the Mathematical Monthly, by whom it was consigned to the waste basket. This was a great humiliation to the honorable gentleman, who, from the high office which he held in Washington, opened up a correspondence with the editor. The outcome of it was a great reduction in its length by the author and a subsequent abbreviation by the editor, which reduced it to four pages, and then it was printed, much to Bartlett's disgust.

On another occasion a professor somewhere in the South sent a paper, much of which was extracted, almost verbatim so far as the formulæ went, from Walton's Problems of Mechanics. The question arose how to let the man know, in the gentlest way, that we had detected the fraud. Bartlett's advice was characteristic: "Just write to him that we don't publish stolen articles." I suggested that it would suffice to let him understand in the gentlest way that we knew where the matter came from. This Runkle did, calling his attention to the similarity of his paper to the book of Walton, with which the professor was probably not acquainted. The answer showed that we had mistaken our man. He assured us that he had a copy of Walton and knew all about the matter, and did not think the resemblance was any objection to his paper. There was an added intimation that, while he should be glad to write for the Journal, he had no time to spend in writing rejected articles.

It was soon found that the conduct of the Journal was beset with even greater difficulties than those of dealing with would-be contributors. Unsustained by any public body, it needed a good list of subscribers to be self-supporting. Between the highest grade of mathematics that could be understood by a sufficient number of professors and students in our country at that time, and the lowest that would be of interest to the professional mathematician, there was a very wide gap. In trying to fill it, complaints were heard on the one side that the papers were too difficult to be understood, and, on the other side, that the standard was too low for a journal which was really to promote the advancement of mathematics.

Yet another question was whether it would be allowable to go outside its proper field for the sake of enlarging its sphere of readers. In one of its early numbers was published a paper by Geo. P. Bond on Donati's Comet which, being the first scientific description of that brilliant object to which the public had access, did a great deal to excite interest in the Journal. It thus lost the support of no less a personage than Professor Peirce, who held that the Journal had no right to go outside its proper field.

At the end of the year 1860 three annual volumes had been published. The Civil War was then imminent and public conditions such that the thought of continuing the enterprise had to be abandoned. The historic value of the publication, I think, can not be doubted. Its contents were more varied and interesting, and low though its standard might have been when we measure it by that of the foreign journals, in contained more varied matter of interest to the

student, even of today, than any American mathematical journal that had preceded it. Its great mission, which it successfully fulfilled, was to interest our college men in the subject. Although we do not find among its contributors the names of any who, in recent times, have become prominent as mathematical investigators, I believe that the stimulus it gave to the study of the subject may have been an important factor in the serious advance which commenced fifteen years later at the Johns Hopkins and other of our great universities.

THE AYER PAPYRUS.*

Edited by EDGAR J. GOODSPEED.

In the Eygptian Room of the Field Columbian Museum, Chicago, there is a Greek papyrus of much mathematical interest. It preserves a series of geometrical processes, in which the areas of various quadrilaterals are determined by methods much less advanced than those of Euclid. The papyrus measures cm. 21.3 x 40.5. Originally it formed part of a roll, written in clear uncial characters on one side only, the writing being in columns. Parts of three columns remain, the second and third being nearly complete, while of the first there are but a few scattered words. The uniformity of the language used, and the preservation of three out of the four figures illustrating the processes of these two columns make their restoration easy and certain. The papyrus comes from the Fayûm, and was brought from Egypt to the Field Museum by Mr. Ed. E. Ayer, of Chicago, in 1895. It belongs to the first century after Christ, but the methods pursued in the calculations, and the senses in which certain terms are used carry back the probable date of the work into pre-christian times and suggest relationship with the school of Heron of Alexandria. The work was apparently a practical treatise on mensuration, designed for use in re-surveying farm-lands of irregular shape, after the annual inundation.

Geometrical figures illustrating the proceses described are appended to them. These figures are covered by numerals indicative of the length of each side, part of a side, and perpendicular, and of the area of each section. It will be observed that in the papyrus these figures are very roughly drawn, and by no means satisfy the proportions thus prescribed for them. For the restoration of the second figure, which is missing from the papyrus through mutilation at that point, I am indebted to Professor E. H. Moore. It should be noted that the word translated "acre" is the Greek "aroura," which is strictly equivalent to about two-thirds of the English acre.

The processes, literally translated, are as follows:

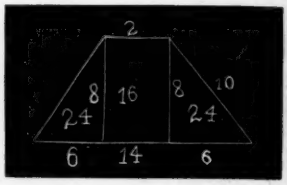
*For the Greek text see Edgar J. Goodspeed, *American Journal of Philology*, Vol. XIX, pp. 25-39.

Column I (restored).

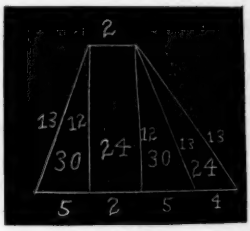
[If there be given an isosceles trapezoid such as the one drawn below, according to the conditions of the problem, the 10 squared=100, and the 2 of the upper side from the 14 of the base leaves 12, ½ of which is 6. This squared=36. Subtract this from 100; the remainder is 64, of which the square root is 8, which is the length of the perpendicular. ½ of this=4, and this by the 6 of the base=24; of so many acres is each of the right-angled triangles. And the 8 of the perpendicular by the 2 of the base=16; of so many]

Column II.

acres is the rectangle in it. Altogether, 64 acres. And the figure will be as follows.



If there be given a scalene trapezoid such as the one drawn below, according to the conditions of the problem, the 13 squared=169, and the 15 squared=225. Subtract the 169; the remainder is 56. Subtract the 2 of the upper side from the 16 of the base; the remainder is 14. Take $\frac{1}{14}$ of 56; it is 4. This from the 14 of the base leaves 10. ½ of this leaves 10. ½ of this=5. This squared=25. Subtract this from the 169; the remainder is 144, of which the square root is 12, which is the length of the perpendicular. This by the 5 of the base=60, ½ of which is 30; of so many acres is each of the right-angled triangles. And the 12 (multiplied) by the 2 of the upper side=24; of so many acres is the rectangle in it. And the 12 multiplied by the 4 of the base=48, ½ of which is 24; of so many acres is the obtuse-angled triangle in it. Altogether, 108 acres. And the figure will be as follows.

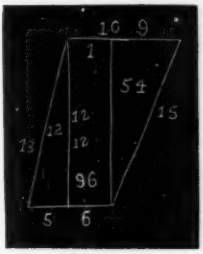


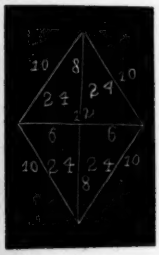
[Figure restored].

Column III.

If there be given a perallelogram such as the one drawn below, according to the conditions of the problem, the 13 of the side squared=169, and the 15 of the side squared=225. Subtract 169 from this; the remainder is 56. Subtract the 6 of the base from the 10 of the upper side; the remainder is 4. Take ¼ of 56; it is 14. Subtract the 4; the remainder is 10, ½ of which is 5, which is the length of the base of the right-angled triangle. This squared=25. And the 13 squared=169. Subtract the 25; the remainder is 144, the square root of which is 12, which is the length of the perpendicular. And subtract the 5 from the 6

of the base; the remainder is one. The 1 from the 10 of the upper side leaves 9, which is the length of the remainder of the upper base, (which remainder is the base) of the right-angled triangle. And the 12 of the perpendicular by the 5 of the base=60, ½ of which is 30; of so many acres is the right-angled triangle in it. And the 12 by the 1=12; of so many acres is the rectangle in it. And 12 by the 9 of the base=108, ½ of which is 54; of so many acres is the other right-angled triangle. Altogether, 96 acres. And the figure will be as follows:





If there be given a rhomb such as the one drawn below, according to the conditions of the problem, the 10 squared=100, and ½ of the 12 of the base is 6. This squared=36. Subtract this (from 100); the remainder is 64, of which the square root is 8, which is the length of the perpendicular. This by the (6) of the base=48, ½ of which is 24; of so many acres is each of the right-angled triangles. Altogether, 96 acres.

And the figure will be as follows.

NOTE.—In order to economize space, the figures are set in the paragraphs to which they belong instead of following them. F.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A teacher's monthly salary after $m=2$ increases of $p=20$ and $q=10\%$, is $\$M=\$120$. What was the original salary?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and G. W. GREENWOOD, A. B., McKendree College, Lebanon, Ill.

$$\text{Original salary}=\$M/[(1+p)(1+g)\text{.....to } m \text{ terms})$$
$$=\$120/[(1.10)(1.20)]$$
$$=\$120\times\tfrac{10}{11}\times\tfrac{5}{6}=\$90\tfrac{10}{11}.$$

ALGEBRA.

173. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $\sqrt{(a+x+y)}=x$.....(1), $\sqrt{(b+y+z)}=x$.....(2), $\sqrt{(c+z+x)}=x$.....(3).

Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

From (2), (3), and (1), we have

$$x^2-y-z=b.....(4),$$
$$y^2-z-x=c.....(5),$$
$$z^2-x-y=a.....(6).$$

From (4) and (5); from (5) and (6); from (6) and (4), we have

$$(x-y)(x+y+1)=b-c.....(7),$$
$$(y-z)(y+z+1)=c-a.....(8),$$
$$(z-x)(z+x+1)=a-b.....(9).$$

By addition of corresponding members of (7), (8), (9), we get

$$(x-y)(x+y+1)+(y-z)(y+z+1)+(z-x)(z+x+1)=0.....(10).$$

It is easy to see that either $x=y=z$.....(11),
or $x+y+1=y+z+1=z+x+1=0$.....(12),

will satisfy (10). From (11), (4), (5), (6), we find

$$x=y=z=1\pm\sqrt{(b+1)},\ 1\pm\sqrt{(c+1)},\ 1\pm\sqrt{(a+1)}.$$

From (4), (5), (6), (12), we obtain

$$x=\pm\sqrt{(b-1)},\ y=\pm\sqrt{(c-1)},\ z=\pm\sqrt{(a-1)}.$$

Also solved by *MARCUS BAKER*, and *G. B. M. ZERR*.

174. Proposed by HARRY S. VANDIVER, Bala, Pa.

If the quantity x be expressed in the form of a continued fraction P_n/Q_n denoting the $(n+1)$th convergent, with x_n the corresponding complete quotient, then $\dfrac{P_{n-(k-1)}-Q_{n-(k+1)}x}{P_n-Q_nx}=(-1)^{k+1}x_n\times x_{n-1}.....x_{n-k}$.

Solution by G. B. M. ZERR, A. M., Ph.D., The Temple College, Philadelphia, Pa., and J. E. SANDERS, Hackney, Ohio.

$$x=\frac{x_nP_n+P_{n-1}}{x_nQ_n+Q_{n-1}} \text{ or } x_n=\frac{xQ_{n-1}-P_{n-1}}{P_n-xQ_n}.$$

Similarly,

$$x_{n-1}=\frac{xQ_{n-2}-P_{n-2}}{P_{n-1}-xQ_{n-1}},\ x_{n-2}=\frac{xQ_{n-3}-P_{n-3}}{P_{n-2}-xQ_{n-2}},\ \ldots\ x_{n-k}=\frac{xQ_{n-(k+1)}-P_{n-(k+1)}}{P_{n-k}-xQ_{n-k}}.$$

$$\therefore x_n x_{n-1}\ldots x_{n-k}=\left[\frac{xQ_{n-1}-P_{n-1}}{P_n-xQ_n}\right]\cdot\left[\frac{xQ_{n-2}-P_{n-2}}{P_{n-1}-xQ_{n-1}}\right]\cdot\left[\frac{xQ_{n-3}-P_{n-3}}{P_{n-2}-xQ_{n-2}}\right]\cdot$$

$$\left[\frac{xQ_{n-4}-P_{n-4}}{P_{n-3}-xQ_{n-3}}\right]\ldots\ldots\left[\frac{xQ_{n-(k+1)}-P_{n-(k+1)}}{P_{n+k}-xQ_{n-k}}\right]$$

$$=(-1)^{k+1}\left[\frac{P_{n-(k+1)}-xQ_{n-(k+1)}}{P_n-xQ_n}\right]=(-1)^{k+1}(A),\ \text{suppose.}$$

$$\therefore (-1)^{k+1}x_n\times x_{n-1}\ldots x_{n-k}=(-1)^{2k+2}(A)=A=\text{result stated.}$$

Also solved in the same manner by *G. W. GREENWOOD.*

GEOMETRY.

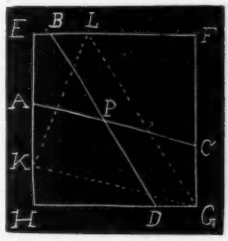
195. Proposed by F. L. SAWYER, Mitchell, Ontario, Canada.

The diagonals of a four-sided figure are h and k, and the area is A; show that the area of the circumscribing square is

$$\frac{h^2k^2-4A^2}{h^2+k^2-A}.$$

Solution by J. R. HITT, Principal Liberty High School, Goss, Miss.; J. SCHEFFER, A. M., Hagerstown, Md.; and L. L. LOCKE, Professor of Mathematics, Adelphos College, Brooklyn, N. Y.

Let $AC=h$, $BD=k$, be the diagonals of the four-sided figure, and $EFGH$ the circumscribing square. Draw GK, GL, parallel to h, k, respectively, and denote HK, FL, by a, b, respectively.



$\triangle GKL=\frac{1}{2}GK.GL\sin KGL=\frac{1}{2}hk\sin P=A$. If $x=$ side of square, we have, $\triangle GKL=A=x^2+\frac{1}{2}ax-\frac{1}{2}bx-\frac{1}{2}(a+x)(x-b)=\frac{1}{2}(x^2+ab)$.

Hence, $b=1/a(2A-x^2)$.....(1).

Also, $GL^2=k^2=x^2+b^2$, $GK^2=h^2=x^2+a^2$.

Hence, $a^2=h^2-x^2$.....(2), and, by adding, we get $2x^2=h^2+k^2-(a^2+b^2)$.....(3). Substituting in (3) the values of b and a from (1), (2), we have, $2x^2=(h^2+k^2)-(2A-x^2)^2/(h^2-x^2)-(h^2-x^2)$, whence, $x^2(h^2+k^2-4A)=h^2k^2-4A^2$.

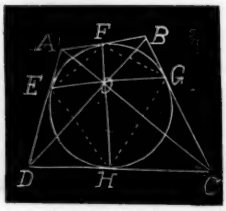
Therefore, $x^2=\dfrac{h^2k^2-4A^2}{h^2+k^2-4A}$.

Also solved by *G. B. M. ZERR.*

196. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a quadrilateral circumscribe a circle, the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

I. Solution by J. R. HITT, Principal, Liberty High School, Goss. Miss.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.



Let the diagonals of $ABCD$ meet in O, and let EG, FH meet in O'. FG, EH are the polars of B, D, respectively. Hence, BD is the polar of the intersection of FG, EH. Likewise is AC the polar of intersection of EF, GH. Therefore, O is the pole of third diagonal of $EFGH$. But O' is the pole of the third diagonal of $EFGH$, since the diagonal triangle is self-conjugate with respect to circumscribing circle. Hence O, O', coincide, and the proposition is proved.

II. Solution by G. W. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon, Ill.

We can project the given quadrilateral into a parallelgram circumscribing an ellipse, in which it is easily seen that the lines joining the points of contact of the opposite sides, and the diagonals of the parallelogram all meet in the center, and that these lines in the original figure are concurrent.

Also solved by *L. C. WALKER*, and *L. L. LOCKE*.

CALCULUS.

161. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Texas.

A cylindrical oil tank of length l and radius r is capped by curved ends and rests with the axis horizontal. The total length of the tank is $l+2h$. If the oil stands at depth d in the tank (d less than $2r$) find its volume (a) when the ends are portions of the surface of a sphere, (b) when the ends are portions of the surface of an ellipsoid.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $y^2+z^2=2ry$, be the equation to the cylinder. Then $z=\sqrt{(2ry-y^2)}$.

$$\therefore V=2l\int_0^d \sqrt{(2ry-y^2)}\,dy=\frac{l}{2}\left[\pi r^2+2(d-r)\sqrt{(2rd-d^2)}+2r^2\sin^{-1}\left(\frac{d-r}{r}\right)\right].$$

If $d=2r$, $V=\pi r^2 l$.

(a). Let $x^2+y^2+z^2=R^2=[(r^2+h^2)/2h]^2$, be the equation to a sphere, a portion of whose surface forms the ends of the tank.

$\therefore z=\sqrt{(R^2-y^2-x^2)}$, the limits of x are $(R-h)$ and $\sqrt{(R^2-y^2)}$; of y, $-r$ and $d-r$. Let V_1=the volume of both ends.

$$V_1 = 4\int_{-r}^{d-r}\int_{R-h}^{\sqrt{(R^2-y^2)}} \sqrt{(R^2-y^2-x^2)}\,dy\,dx$$

$$= \int_{-r}^{d-r}\Big[\pi(R^2-y^2) - 2(R-h)\sqrt{(2Rh-h^2-y^2)}$$

$$-2(R^2-y^2)\sin^{-1}\Big(\frac{R-h}{\sqrt{(R^2-y^2)}}\Big)\Big]dy.$$

Integrating, substituting the value of R, and reducing,

$$V_1 = \frac{d-r}{12h^2}\Big(3(r^2+h^2)^2 - 4h^2(d-r)^2\Big)$$

$$\times\Big[\pi - 2\sin^{-1}\Big(\frac{r^2-h^2}{\sqrt{[(r^2-h^2)^2+4dh^2(2r-d)]}}\Big)\Big]$$

$$+\tfrac{1}{6}\pi h(3r^2+h^2) - \frac{(d-r)(r^2-h^2)\sqrt{(2rd-d^2)}}{3h}$$

$$+\frac{(r^2+h^2)^3}{6h^3}\tan^{-1}\Big[\frac{(d-r)(r^2-h^2)}{(r^2+h^2)\sqrt{(2rd-d^2)}}\Big]$$

$$-\frac{r^2-h^2}{6h^3}(r^4+4r^2h^2+h^4)\sin^{-1}\Big(\frac{d-r}{r}\Big).$$

If $d=2r$, $V_1=\frac{1}{3}\pi h(3r^2+h^2)$. If $d=2r$, $h=r$, $V_1=\frac{4}{3}\pi r^3$.

Total volume$=V+V_1$.

(b). Let $x^2/a^2+(y^2+z^2)/b^2=1$, be the equation to the ellipsoid. It is necessary to know a or b.

If a is known, $b=ar/\sqrt{(2ah-h^2)}$.

$z=b\sqrt{[1-(y^2/b^2)-(x^2/a^2)]}$, the limits of x are $(a-h)$ and $a/b\sqrt{(b^2-y^2)}$; of y, $-r$ and $d-r$.

Let V_2=volume of both ends.

$$V_2 = 4b\int_{-r}^{d-r}\int_{a-h}^{a/b\sqrt{(b^2-y^2)}} \sqrt{[1-(y^2/b^2)-(x^2/a^2)]}\,dy\,dx,$$

$$= \int_{-r}^{d-r}\Big[\frac{\pi a}{b}(b^2-y^2) - \frac{2(a-h)}{a}\sqrt{[b^2(2ah-h^2)-a^2y^2]}$$

$$-\frac{2a}{b}(b^2-y^2)\sin^{-1}\Big(\frac{b(a-h)}{a\sqrt{(b^2-y^2)}}\Big)\Big]dy.$$

Integrating, substituting value of b, and reducing,

$$V_2=\frac{(d-r)[3a^2r^2-(d-r)^2(2ah-h^2)]}{3r\sqrt{(2ah-h^2)}}$$

$$\left[\pi-2\sin^{-1}\left(\frac{r(a-h)}{\sqrt{[a^2r^2-(d-r)^2(2ah-h^2)]}}\right)\right]$$

$$+\frac{\pi r^2h(3a-h)}{3(2a-h)}-\tfrac{4}{3}(d-r(a-h)\sqrt{(2dr-d^2)}$$

$$+\frac{4a^3r^2}{3(2ah-h^2)}\tan^{-1}\left(\frac{(a-h)(d-r)}{a\sqrt{(2rd-d^2)}}\right)-\frac{2r^2(a-h)\ (2a^2+2ah-h^2)}{3(2ah-h^2)}\sin^{-1}\left(\frac{d-r}{r}\right).$$

If $d=2r$, $V_2=\dfrac{2\pi r^2h(3a-h)}{3(2a-h)}$.

If $d=2r$, $a=h$, $V_2=\frac{4}{3}\pi r^2h$. Total volume$=V+V_2$.

Also solved by *G. W. GREENWOOD.*

162. Proposed by **J. E. SANDERS**, Hackney, O.

Solve the differential equations

$$(a)\ x\frac{dy}{dx}-y=x\sqrt{(x^2+y^2)},\quad (b)\ \cos x\frac{dy}{dx}+y=1-\sin x.$$

Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.; **E. L. SHERWOOD**, Professor of Mathematics, Shady Side Academy, Pittsburg, Pa.; **M. E. GRABER**, Graduate Student, Heidelberg University, Tiffin, O.; and the **PROPOSER**.

(a) $xdy-ydx=x[x^2+y^2]dx$. Let $y=vx$. $\therefore\ dy=vdx+xdv$.

$$\therefore\ dx=\frac{dv}{\sqrt{[1+v^2]}}.\quad \therefore\ x+A=\log\{v+\sqrt{[1+v^2]}\}.$$

$$\therefore\ x+A=\log\left(\frac{y+\sqrt{[x^2+y^2]}}{x}\right).$$

(b) $\cos x dy+ydx=[1-\sin x]dx$. Let $y=v[1-\sin x]$.

$$\therefore\ dy=[1-\sin x]dv-v\cos x dx.\quad \therefore\ dx=\cos x dv-v\sin x dx=d[v\cos x].$$

$$\therefore\ x+A=v\cos x=\frac{y\cos x}{1-\sin x}.\quad \therefore\ y\cos x=[x+A][1-\sin x].$$

Solved in the same way by *HOMER R. HIGLEY, LON C. WALKER*, and *J. SCHEFFER.*

MECHANICS.

153. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

An equiangular polygon consisting of equal, freely jointed rods, is hung up from vertex, A. The vertices adjacent to A are connected by a light rod of such length that the polygon is still regular. Find the stress in the rod and the reactions at the vertices.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be the point of suspension, O, the centroid of the system, T, the stress on the weightless rod FB, nW, the weight of the rods, $FB=x$, $AB=a$, $AO=y$, $\angle OAB=\angle OBA=\theta$, R=reaction.

Suppose the deformation such that the centroid moves vertically through a small space. Then $Tdx-nWdy=0$.

$BG : AB=\sin\theta : 1$; $\therefore\ x=2a\sin\theta$.

$AO : AB=\sin\theta : \sin(\pi-2\theta)$; $\therefore\ y=a\sin\theta/\sin(\pi-2\theta)$.

$\therefore\ y=a\sin\theta/\sin2\theta=\frac{1}{2}a\sec\theta$.

$\therefore\ dx=2a\cos\theta d\theta$, $dy=\frac{1}{2}a\sec\theta\tan\theta d\theta$.

$\therefore\ 2aT\cos\theta d\theta=\frac{1}{2}(nWa)\sec\theta\tan\theta d\theta$.

$\therefore\ T=\frac{1}{4}nW\sec^2\theta\tan\theta$. R acts along BO, $\angle ABG=\frac{1}{2}\pi-\theta$.

$\therefore\ \angle OBG=\angle OFG=2\theta-\frac{1}{2}\pi$, $\angle BOF=2\pi-4\theta$.

$\therefore\ R : T=\sin(2\theta-\frac{1}{2}\pi) : \sin(2\pi-4\theta)$; $R : T=\cos2\theta : \sin4\theta$.

$\therefore\ R=T\cos2\theta/\sin4\theta=T/2\sin2\theta=T/4\sin\theta\cos\theta$. $\therefore\ R=\frac{1}{16}nW\sec^4\theta$.

Also solved by G. W. GREENWOOD.

154. Proposed by M. E. GRABER, Graduate Student, Heidelberg University, Tiffin, Ohio.

Find the form of the curve in a vertical plane, such that a heavy bar resting on its concave side and on a peg at a given point, (the origin), may be at rest at all positions.

Solution by G. W. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon, Ill.; and E. L. SHERWOOD, Professor of Mathematics, Shady Side Academy, Pittsburg, Pa.

Let the bar be homogeneous and of length $2b$. Assume the peg and the curve to be smooth. Take the peg as origin, the horizontal line through it in the plane as initial line, and measure θ downwards.

The coördinates of the lower extremity being (r, θ), those of the center of the rod are $(r-b, \theta)$. Since in all positions the rod is in equilibrium, the center is at a constant distance from the initial line. Hence $(r-b)\sin\theta=a$, is the curve traced by one extremity.

Also solved by G. B. M. ZERR.

DIOPHANTINE ANALYSIS.

107. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Required the least three positive integral numbers such that the sum of all three of them and the sum of every two of them shall be a square number.

II. Solution by CHARLES C. CROSS, Whaleyville, Va.

Consider the problem in its most general sense: Required the least n positive integral numbers such that the sum of all of them, and the sum of every $n-1$ of them, shall be squares.

Let $x_1, x_2, x_3, \ldots, x_n$ be the numbers; then

$$x_1+x_2+x_3+\ldots+x_n=a^2\text{(say)}\ldots(1);$$
$$x_1+x_2+x_3+\ldots+x_{n-1}=a_1^2\text{(say)}\ldots(2);$$
$$x_1+x_2+x_3+\ldots+x_{n-2}=a_2^2\text{(say)}\ldots(3);$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots+x_{n-2}+x_{n-1}+x_n=a^2{}_n\text{(say)}\ldots(n).$$

From (2), (3), ..., (n) we find

$$x_1=\frac{1}{n-1}[a_1^2+a_2^2+a_3^2\ldots-(n-2)a^2{}_n]\ldots(4).$$

$$x_2=\frac{1}{n-1}[a_1^2+a_2^2-(n-2)a_3^2\ldots+a^2{}_n]\ldots(5).$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$x_n=\frac{1}{n-1}[-(n-2)a_1^2+a_2^2+a_3^2\ldots+a^2{}_n]\ldots(6),$$

where $x_1, x_2, \ldots, x_n$ and a are connected by the relation

$$a_1^2+a_2^2+a_3^2+\ldots+a^2{}_n=(n-1)a^2\ldots(7).$$

In (7), let $a_2=a-m_1$; $a_3=a-m_2$; $a_4=a-m_3$; .... $a_n=a-m_{n-1}$; whence

$$a=\frac{a_1^2+m_1^2+m_2^2+\ldots+m^2{}_{n-1}}{2[m_1+m_2+\ldots+m_{n-1}]}.$$

From which we readily find,

$$a_2=\{a_1^2-m_1^2+m_2^2+\ldots+m^2{}_{n-1}-2[m_1m_2+m_1m_3+\ldots+m_1m_{n-1}]\}$$
$$\div 2[m_1+m_2+\ldots+m_{n-1}],$$
$$a_3=\{a_1^2+m_1^2-m_2^2+\ldots+m^2{}_{n-1}-2[m_1m_2+m_2m_3+\ldots+m_2m_{n-1}]\}$$
$$\div 2[m_1+m_2+\ldots+m_{n-1}].$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$a_n=\{a_1^2+m_1^2+m_2^2+\ldots-m^2{}_{n-1}-2[m_1m_{n-1}+m_2m_{n-1}+\ldots+m_{n-2}m_{n-1}]\}$$
$$\div 2[m_1+m_2+\ldots+m_{n-1}].$$

These values in (7) give, after reducing,

$$4[m_1+m_2+\ldots+m_{n-1}]^2a_1^2+\{a_1^2-m_1^2+m_2^2+\ldots+m^2{}_{n-1}-2[m_1m_2+m_1m_3+\ldots+m_1m_{n-1}]\}^2+\{a_1^2+m_1^2-m_2^2+\ldots+m^2{}_{n-1}-2[m_1m_2+m_2m_3+\ldots+m_2m_{n-1}]\}^2+\ldots+\{a_1^2+m_1^2+m_2^2+\ldots-m^2{}_{n-1}-2[m_1m_{n-1}+m_2m_{n-1}+\ldots+m_{n-2}m_{n-1}]\}^2=[n-1][a_1^2+m_1^2+m_2^2+\ldots+m^2{}_{n-1}]^2\ldots\ldots(8).$$

In (8), let a_1^2, a_2^2.....$a^2{}_n$ represent the first, second.....and nth expressions of the left hand member and a that of the right.

These values of a_1^2, a_2^2.....$a^2{}_n$ in (4), (5).....(6) give the corresponding values of x_1, x_2.....x_n.

In the problem, $n=3$; whence

$$a^2=[a_1^2+m_1^2+m_2^2]^2,\ a_1^2=4[m_1+m_2]^2a_1^2;$$
$$a_2^2=[a_1^2+m_2^2-m_1^2-2m_1m_2]^2;\text{ and}$$
$$a_3^2=[a_1^2+m_1^2-m_2^2-2m_1m_2]^2.$$

These values of a_1^2, a_2^2 and a_3^2 in (4), (5).....and (6) give

$$x_1=4[a_1^2m_2^2+a_1^2m_1m_2-m_1m_2^3+m_1^3m_2];$$
$$x_2=4[a_1^2m_1^2+a_1^2m_1m_2+m_1m_2^3-m_1^3m_2];\text{ and}$$
$$x_3=a_1^4+m_1^4+m_2^4+2m_1^2m_2^2-2a_1^2m_1^2-2a_1^2m_2^2-8a_1^2m_1m_2.$$

Let $a_1=2$, $m_1=3$, and $m_2=4$; then $x_1=112$, $x_2=672$, and $x_3=57$.
$\therefore x_1+x_2+x_3=29^2$; $x_1+x_2=28^2$; $x_1+x_3=13^2$; and $x_2+x_3=27^2$.

An excellent solution was also received from *A. H. BELL*.

109. Proposed by HARRY S. VANDIVER, Bala, Pa.

If $m+n+1$ is a prime integer, show that $m!\times n!-(-1)^{\frac{1}{2}(3m-n)}$ is divisible by $m+n+1$. For instance, $6!\times4!-(-1)^7$ is divisible by 11.

Correction and Solution by DR. L. E. DICKSON, The University of Chicago.

In the formula proposed, the exponent of -1 is not correct. Thus, for $m=5$, $n=5$, the number $5!\times5!-(-1)^5$ is not divisible by 11, whereas $5!\times5!-(-1)^4$ is divisible by 11. The correct formula may be stated thus:

$$m!\times(p-1-m)!\equiv(-1)^{m-1}\ (\mathrm{mod}p),$$

where p is any odd prime and m any integer $<p$.

The proof follows from the congruences

$$(p-1)!\equiv-1(\mathrm{mod}p),\qquad\frac{(p-1)!}{m!(p-1-m)!}\equiv(-1)^m\ (\mathrm{mod}p),$$

of which the first is Wilson's Theorem, while the left member of the second is the coefficient of $x^m y^{p-1-m}$ in $(x+y)^{p-1}$. The latter is congruent, modulo p, to

$$\frac{x^p+y^p}{x+y} = x^{p-1} - x^{p-2}y + \ldots + (-1)^m x^m y^{p-1-m} + \ldots + y^{p-1}.$$

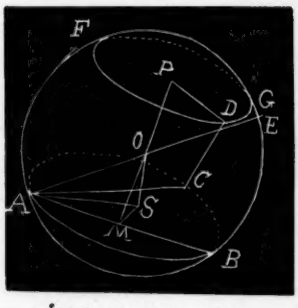
AVERAGE AND PROBABILITY.

130. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three and three, is 1-35 of the volume of the given sphere.

I. Solution by the PROPOSER.

Choose A, B, C, D as the four random points; O the center of the given sphere with radius r; $ABFE$ a great circle through A, B; ABC a small circle through A, B, C, with center S; DGF a small circle through D parallel to ABC, with center P; M the middle point of AB.



Put $OP=x$, $AS=r_1$, $\angle AOB=\theta$, $\angle OMS=\phi$, $\angle CAB=\psi$, $\angle SAM=\psi_1$. Then we have

$AM=r\sin\frac{1}{2}\theta=r_1\cos\psi_1$,

$SM=r\cos\frac{1}{2}\theta\cos\phi=r_1\sin\psi_1$,

$OS=r\cos\frac{1}{2}\theta\sin\phi$,

$AC=2r_1\cos(\psi-\psi_1)$,

$PD=\sqrt{(r^2-x^2)}$,

$r_1=r(\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta\cos^2\phi)^{\frac{1}{2}}$, area $ABC=2\pi r_1\sin\frac{1}{2}\theta\sin\psi\cos(\psi-\psi_1)$,

volume of tetrahedron $D\text{—}ABC=\frac{1}{3}SP.\text{area}\,ABC=\frac{2}{3}\pi r r_1(x+r\cos\frac{1}{2}\theta\sin\phi)\sin\frac{1}{2}\theta \times\sin\theta\cos(\psi-\psi_1)$.

Hence, we have for the required average volume, $V=\frac{4}{105}\pi r^3=\frac{1}{35}$ of the volume of the given sphere.

[For the integration, see solution in last issue, page 113, where the figure for Professor Zerr's solution was inserted for the one belonging to Professor Walker's solution. F.]

131. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

A sphere is described with its center within a given sphere, and its surface intersecting the surface of the given sphere. The average volume common to both spheres is 10/21 of the volume of the given sphere.

Solution by the PROPOSER.

Let M be the center of the given sphere, and N that of the random sphere.

Put $BM=a$, $NB=x$, $MN=y$, $\angle MBN=\theta$, $\angle BMN=\phi$, $\angle BNM=\psi$.

Then we have

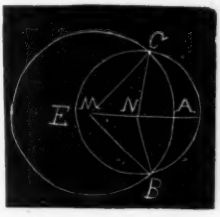
$$\cos\phi=\frac{1}{y}(a-x\cos\theta),$$

$$\cos\psi=\frac{1}{y}(x-a\cos\theta),$$

$$y^2=a^2+x^2-2ax\cos\theta.$$

Volume of spherical section $BACM=\frac{2}{3}\pi a^3(1-\cos\phi)=\frac{2}{3}\pi[a^3-1/y(a^4-a^3\cos\theta)]$;
volume of spherical sector $BECN=\frac{2}{3}\pi x^3(1-\cos\psi)=\frac{2}{3}\pi[x^3-1/y(x^4-ax^3\cos\theta)]$;
volume of solid $BMCN=\frac{1}{3}\pi a^2 y\sin^2\phi=\frac{1}{3}\pi(a^2x^2/y)\sin^2\theta$;
volume of solid $BACE=S=\frac{1}{3}\pi[2a^3+2x^3-2/y\ (a^4+x^4)+2/y\ (a^3x+ax^3)\cos\theta-(a^2x^2/y)\sin^2\theta]$.

Hence we have for the required average volume,



$$V=\frac{\int_0^a\int_{a-x}^{a}S.4\pi y^2\,dxdy+\int_a^{2a}\int_{x-a}^{a}S.4\pi y^2\,dxdy}{\int_0^a\int_{a-x}^{a}4\pi y^2\,dxdy+\int_a^{2a}\int_{x-a}^{a}4\pi y^2\,dxdy}$$

$$=\frac{2\pi}{3a^4}\left\{\int_0^a\int_{a-x}^{a}\left[2y^2(a^3+x^3)-2y(a^4+x^4)+2y(a^3x+ax^3)\cos\theta-a^2x^2y\sin^2\theta\right]dxdy\right.$$

$$\left.+\int_0^{2a}\int_{x-a}^{a}\left[2y^2(a^3+x^3)-2y(a^4+x^4)+2y(a^3x+ax^3)\cos\theta-a^2x^2y\sin^2\theta\right]dx\,dy\right\}$$

$$=\frac{2\pi}{3a^4}\left\{\int_0^{2a}\left[\tfrac{8}{3}a^6-2a^5x+2a^4x^2+\tfrac{5}{4}a^2x^4-2ax^5+\tfrac{1}{2}\tfrac{7}{4}x^6\right]dx\right.$$

$$\left.+\int_0^a\tfrac{8}{3}(a^3+x^3)(x-a)^3dx+\int_a^{2a}\tfrac{8}{3}(a^3+x^3)(a-x)^3dx\right\}$$

$=\frac{4}{6}\frac{0}{3}\pi a^3=\frac{1}{2}\frac{0}{1}$ of the volume of the given sphere.

Also solved by *G. B. M. ZERR.*

132. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

n points are taken at random on the circumference of a given circle. Prove that the chance of the center of the circle lying within the polygon formed by joining these points is $1-(1/2^{n-2})$.

Solution by the PROPOSER.

Using a figure similar to the one used in problem 116, let $AOD=\theta_1$, $BOD=\theta_2$, $COD=\theta_3$, etc. Let p be the required chance, p_1, the chance that it does not contain the center of the circle.

$$p_1=\frac{2\int_0^{\frac{1}{2}\pi}\int_0^{\theta_1}\int_0^{\theta_2}\cdots\int_0^{\theta_{n-2}} d\theta_1 d\theta_2 d\theta_3 \cdots d\theta_{n-1}}{\int_0^{\pi}\int_0^{\theta_1}\int_0^{\theta_2}\cdots\int_0^{\theta_{n-2}} d\theta_1 d\theta_2 d\theta_3 \cdots d\theta_{n-1}}=\frac{(n-1)!}{\pi^{n-1}}\cdot\frac{2(\frac{1}{2}\pi)^{n-1}}{(n-1)!}=\frac{1}{2^{n-2}}.$$

$$p=1-p_1=1-(1/2^{n-2}).$$

Also solved with the same result by J. SCHEFFER.

MISCELLANEOUS.

133. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a group G of order mn has a subgroup H of order n, and if n has no prime factor which is less than m, show that H must be a self-conjugate subgroup. (Frobenius.)

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let the substitution of the given subgroup (H) be $1, t_2, t_3, \ldots t_p$.

If H is not self-conjugate, multiply it into any substition (S) which is not commutative to it. If the transform (H_1) of (H) with respect to any one of these products contains a substitution $(s^{-1}ts)$ of order m^β $(\beta>0)$ which is not found in H and if t does not transform H_1 into itself we can form the following rectangle with m conjugate rows of p elements:

$$\begin{array}{llll} s, & st_2, & st_3, & \ldots st_p \\ ts, & tst_2, & tst_3, & \ldots tst_p \\ st^2, & t_2st^2, & t_3st^2, & \ldots t_pst^2 \\ \ldots & \ldots & \ldots & \\ st^{m-1}, & t_2st^{m-1}, & t_3st^{m-1}, & \ldots t_pst^{m-1}. \end{array}$$

All the substitutions of a given row transform H into the same group, while any two substitutions from different rows transform H into two different groups. As no substitutions in the given rectangle can be equal to each other, the entire group would have to contain at least $p(m+1)$ substitutions. This is contrary to the hypothesis. Therefore H is a self-conjugate sub-group.

134. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Give a complete solution of the Jacobian equation $\kappa^2\mathrm{sn}^4u+2\kappa^2\mathrm{sn}^2u+1=0$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\text{sn}u=x$. $\therefore x^4+2x^2+1/\kappa^2=0$, where κ is the modulus.

$\therefore (x^2+1)^2=-(1-\kappa^2)/\kappa^2$.

$$\therefore x=\pm\sqrt{\frac{-\kappa\pm\sqrt{(-1)}\sqrt{(1-\kappa^2)}}{\kappa}}=\pm n, \text{ suppose.} \quad \therefore \text{sn}u=\pm n.$$

$$\therefore u=2\int_0^n \frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}=2F(\kappa, n)$$

$$=2[n+\tfrac{1}{6}n^3(1+\kappa^2)+\tfrac{1}{40}n^5(3+2\kappa^2+3\kappa^4)+\ldots\ldots].$$

When $\kappa=1$, $u=2\sqrt{-1}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots)$, since $n=\pm\sqrt{-1}$).

$\therefore u=\frac{1}{2}\pi\sqrt{-1}$.

But $u=2\int_0^{\sqrt{-1}} \frac{dx}{1-x^2}=\log\left(\frac{1+\sqrt{-1}}{1-\sqrt{-1}}\right)$.

$\therefore \pi=2\sqrt{-1}\log\left(\frac{1-\sqrt{-1}}{1+\sqrt{-1}}\right)$, a result previously referred to in this journal.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

167. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A traveler notices that $m=2\frac{1}{2}$ times the number of spaces between the telegraph poles that he passes in a minute is the rate of train in miles per hour. How far apart are the poles?

168. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If I buy for $m=10\%$ and $n=5\%$ *less* I shall gain $p=15\%$ and $q=5\%$ *more*. What is my rate of gain?

ALGEBRA.

178. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A_n being the arithmetic mean of the nth powers of the numbers less than p and prime to it, find a relation between A_3, A_2 and p.

179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left(\frac{q^2}{p}-\frac{p^3}{5q}\right)=0.$$

180. Proposed by the late JOSIAH H. DRUMMOND.

If r/s is such a value of p as makes $m/(p^2-2)$ integral, prove that $(3r+4s)/(2r+3s)$ is another such value, so that an indefinite number of integral values may be obtained.

Also, if r/s is such a value of p as makes $2m/(p^2-2)$ integral, prove that $2(r+s)/(r+2s)$ is also such a value.

GEOMETRY.

201. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Two plane sections of a right circular cone have their major axes AA', aa' coplaner, and Aa on one generator equal to $A'a'$ on the other. The projections of the sections on any plane perpendicular to the axis are confocal.

202. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The equations $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$ and $l\beta\gamma+m\alpha\gamma+n\beta\alpha=0$ represent ellipses. If a, b, c are the sides of the triangle of reference, transform to Cartesian coördinates and find area of each ellipse.

CALCULUS.

166. Proposed by T. N. HAUN, Mohawk, Tenn.

Find the volume of the solid formed by the revolution of the curve $(y^2+x^2)=a^2(x^2-y^2)$ round the axis of x.

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Integrate, $\int_0^a \int_0^b \int_0^c \frac{zdxdydz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$.

MECHANICS.

157. Proposed by T. W. WRIGHT, Schenectady, N. Y.

Explain why a waterfall h feet high can support a column of water $2h$ feet high.

158. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are electrical images of each other in respect to a given sphere may be constant, and that their centers of gravity shall lie on the surface of the sphere.

AVERAGE AND PROBABILITY.

143. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

The extremities of two equal lines drawn from a fixed point in the circumference of a given circle is joined. Find the average area of the circle inscribed in the triangle formed.

144. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

In a circular park 400 feet in diameter are 4 *equal* circular ponds of *variable* diameter. What is the probability that a sightless person walking in a straight line from the center of the park, will step into a pond ?

MISCELLANEOUS.

138. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Find an invarient of the *third degree* in the coefficients of a ternary quartic.

139. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Given the roots of a binary cubic, to find the roots of its two independent covariants.

NOTES.

Professor W. F. Osgood of Harvard University, has been promoted to a full professorship of mathematics. F.

Dr. C. A. Noble has been promoted to an assistant professorship of mathematics at the University of California. F.

Professor Alexander Macfarlane delivered, at Lehigh University, April 20-23, a course of six lectures on the British mathematicians, Kirkham, Babbage, Whewell, Dodgson, Stokes, and Rayleigh. F.

Professor John J. Quinn has brought to public attention a third triangle, to be used with the two triangles commonly used in drawing sets, and in a small circular illustrates many constructions which are easily made by means of this triangle of which he is the inventor. F.

Professor Josiah Willard Gibbs, of Yale University, died at New Haven, April 28th, 1903, of heart disease. Professor Gibbs was born in New Haven, Feb. 11, 1839, and graduated at Yale in 1858. In 1863, he received the degree of Doctor of Philosophy. After studying in Paris, Berlin, and Heidelberg, he was appointed, in 1871, to the Professorship of Mathematical Physics in Yale, which position he held until the time of his death. He was a member of the Royal Society of London, of the National Academy of Science, of the American Mathematical Society, and many other learned bodies. He was an authority of the first rank in thermo-dynamics, and in the application of vector analysis to physical problems. Last year, 1902, he published a work entitled *Elementary Principles in Statistical Mechanics*. F.

BOOKS AND PERIODICALS.

Traité de Géométrie. Per C. Guichard, Professeur L'Université de Clermont, Deuxième Partie, Compléments. Paper bound. Price, 6 fr. Paris: Librarie Nony & C^ie^.

This is Volume II of the Treatise on Geometry. The first volume treats of plane and solid geometry. The second volume is divided into five sections. The first section is divided into nine chapters, and some of the subjects here treated in a very excellent way are, Geometry of Systems of Lines, Systems of Circles, Transversals, Reciprocal Polars, Involution, Inversion, Tangent Circles, and the Nine Point Circle. The second section contains, among other subjects, Orthogonal Projections, Theory of Vectors, and Central Projections. The third part treats of Poles and Polar Planes with Respect to a Sphere, Inverse Figures in Space, Orthogonal Spheres, and Geometry of the Sphere. The fourth part treats of the Conic Sections, and the fifth part of Geometry of Position and the Measurement of Areas. The work contains many interesting theorems and discussions not found in the ordinary text-book on Geometry. F.

Quinn's Geometry Tablet, Adapted to Geometry and Physics. Scranton, Wetmore & Co., Rochester, N. Y.

This Tablet is adapted to formal written work in original demonstrations; a few sheets at the end of the tablet are intended for construction problems. By its use the labor of examining students' daily written work should be materially reduced.

Annals of Mathemathics. Published quarterly, and under the auspices of Harvard University. Price, $2.00 per year, in advance.

The April number contains the following articles: The Cardiod and Tricuspid; Quartics with Three Cusps, by R. C. Archibald; Note on a Partial Differential Equation of the First Order, by Dr. E. D. Roe, Jr.; On a Generalization of the Set of Associated Minimum Surfaces, by A. S. Gale; Twisted Quartic Curves of the First Species, and Certain Covariant Quartics, by H. S. White; On the Characteristics of Differential Equations, Part I, by E. R. Hendrick.

The American Journal of Mathematics. Published, quarterly, under the auspices of the Johns Hopkins University. Price, $5.00 per year in advance.

Numbsr 2, Volume XXV, contains the following articles: The Double Six Configuration Connected with the Cubic Surface, and a Related Group of Cremona Transformations, by Edward Kasner; Untersuchungen über Linear Differentialgleichungen 4 Ordnung und die zugehhörigen Gruppen, Von Saul Epsteen; The Logic of Relations, Logical Substitution Groups, and Cardinal Numbers, by A. W. Whitehead; On Differential Equations Belonging to Ternary Linearoid Groups, by F. E. Ross; On a Certain Group of Isomorphisms, by J. W. Young.

The Mathematical Gazette. Edited by J. W. Greenstreet, Stroud, England.

The May number contains several important articles, among which are On the Representation of Imaginary Points by Real Points in the Plane, and An Elementary Introduction to the Infinitesimal Calculus of Surfaces.

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THE APOLLONIAN PROBLEM IN SPACE.

By EDWARD KASNER.

The so-called Apollonian problem, namely, the construction of a circle tangent to three given circles in a plane, is among the most famous in the domain of elementary geometry. It was proposed (and probably solved) by Apollonius of Perga, in one of his last works, *On Contact;* but the first solution that has come down to us is that given by Vieta in his attempted restoration which he entitled *Gallus Apollonius* (1600).* Since then it has been treated in an almost endless variety of ways, for instance, by Gergonne, Pluecker, Steiner, Hart, Casey, and recently in an elaborate memoir by Study (*Mathematische Annalen*, vol. 49).

The analagous problem in solid geometry is usually taken to be the construction of a sphere tangent to four given spheres.† There is, however, a different extension, which is the subject of this note, namely, the construction of a circle tangent to two given circles in general position in space. That this is a definite problem is seen most simply by an enumeration of constants. A circle in space requires six parameters for its determination, for example, three to fix its plane, two to fix its center in the plane, and one to fix its radius. The number of circles tangent to a given circle is ∞^3; for there are ∞^2 points on the circle, at each of which there are ∞^2 tangent planes, and in each of these planes there are ∞^1 tangent circles. Hence, tangency of circles in space is a three-fold con-

*In response to Vieta's proposal of the problem, a solution was given by Romanus which is however unsatisfactory, as it makes use of conic sections instead of the straight edge and compass only. See Cantor, *Geshichte der Mathematik*, 2nd edition, Vol. II, page 590.

†This was first considered by Fermat in his *De contactibus sphaericis*. See Cantor, Vol. II, page 659.

dition, instead of a simple condition as it is for circles in a plane or spheres in space. If a circle is to be tangent to two given circles, this is equivalent to the imposition of six simple conditions, so that the number of solutions is finite. It will be seen, in fact, that there are four solutions.

Denote the two given circles by C' and C'', their planes by π' and π'', and the line in which the latter intersect by l. The first step in the solution is to construct the sphere S which cuts both C' and C'' othogonally. The center of S must obviously lie in both π' and π'', and hence in l. Moreover, it must be so situated that the tangents from it to C' are equal to those drawn to C''. Hence the center may be obtained as follows: Revolve the circle C'' about the line l as axis until it falls in the plane π'; denote the circles so obtained by F; construct the radical axis of C' and F; this will intersect l in the point P required. For the radical axis is the locus of points for which the tangents drawn to the circles C' and F are equal; and every point in l has the property that the tangents drawn from it to C'' and F are equal. The sphere S has its center at P, and its radius is equal to either of the tangents from P to C' or C''. This construction also shows that S is unique.

Denote the two points in which C' intersects S by A', B'; and similarly the points in which C'' intersects S by A'', B''. Select one point from the first pair, and one from the second pair. Through the two points selected there passes a unique circle* orthogonal to S. This circle will be tangent to both C' and C''. In this way we obtain four circles touching both of the given circles, corresponding to the four possible pairs of points, A', A''; A', B''; B', A''; B' B''.

It remains to be shown that there are no additional solutions of the problem. Let C be any circle tangent toC' and C'' at, say, the points P' and P'' respectively. Construct the tangent lines to C at P' and P'', and denote their point of intersection by Q. With Q as center and radius $QP'=QP''$ describe a sphere. This sphere will be orthogonal to C at P' and P'', hence it is also orthogonal to C' and C''. From what precedes it follows that the sphere constructed coincides with the orthogonal sphere S. Therefore P' must lie at either A' or B', and P'' must lie at either A'' or B'', so that C necessarily coincides with one of the four circles constructed above.

The two given circles and the four circles tangent to them form an interesting set of six circles. To bring out the symmetry of the configuration, it will be convenient to introduce a new notation. Let the four points of intersection on the sphere S be denoted by P_1, P_2, P_3, P_4. Each of the six circles intersects S in two of these points, and conversely to any pair of points there corresponds one of the circles. Denote the circle corresponging to P_i, P_k by C_{ik}. The circles fall naturally into three pairs:

(I)	C_{12}, C_{34};
(II)	C_{13}, C_{24};
(III)	C_{14}, C_{23}.

*To obtain this circle, construct the great circle on S determined by the two points; the required circle then lies in the plane of this great circle and is orthogonal to it at the same two points.